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On a measure of skill for games with chance elements[1]

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Abstract

In various countries, including the Netherlands and Austria, legislation is such[1] that the question whether a specific game should be considered as a game of chance[2] or as a game of skill is predominant in the exploitation decision of private casinos. This paper aims for an objective and operational criterium to quantify the level of skill of casino games in order to establish a mutual ranking. The varions concepts are illustrated by means of variations of Poker.

Keywords: games, skill, chance elements, learnminimax strategy, Poker

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1 Introduction

The commercial exploitation of games with chance elements is a lucrative affair from the point of view of the propietor. In the Netherlands the exploitation of such games is controlled by means of legislation as reflected in the law "Wet op de Kansspelen (WKS)":

a) *.... it is not allowed to: exploit games with monetary prizes if the participants in general do not have a dominant influence on the probability to win, unless in compliance to this act, a license is granted ...*

In practice, the gouvernment only grants such a license to the Holland Casino's foundation. In more countries (e.g. Austria) a similar type of legislation is operative and the questions we address below therefore will arise in some way in other countries than the Netherlands as well.
So, though our analysis will be focussed primarily on the Dutch situation, practical applications certainly will reach further than that. Moreover, from a theoretical point of view, our analysis will offer a general method to measure skill in a broad class of games.

In the remainder we call a game that is covered by the WKS a *game of chance*. A game which is not a game of chance will be called a *game of skill*. So, in particular, to commercialize games of skill no license within the meaning of the WKS is needed. This opens the way for private casinos to compete with the games of chance that are exploited in the Holland Casino's.

Of course the propietor of a game and the legislator can have different views on how to qualify a game since such a judgement about the role of chance is rather subjective and the financial stakes are high. If it would be possible to rank a broad class of games with chance elements by means of an operational and objective criterium which quantifies the level of skill, e.g. on a scale from zero to one, the legislator would be able to decide on a certain bound on the level of skill, below which a game should be considered as a game of chance. In this article we propose a specific definition to measure skill and provide approximations for various casino games. The analysis is partly based on the results in van der Genugten and Borm (1991, 1994). The various concepts will be illustrated in some detail by means of simple variations of the game of Poker. Some general references on specific (computational) aspects on Poker are Epstein (1977), Scarne (1990) and Tamburin (1993).

The paper is organized in the following way.
Section 2 discusses the qualitative requirements we think should be satisfied by a suitable measure of skill based on the existing Dutch jurisprudence.
Then, in Section 3, we propose a definition of the level of skill for so-called "basic" games with a relatively simple structure. This structure enables us to highlight the essential features of the definition. Since basic games hardly occur in reality this section is of a didactic nature. In the subsequent sections the definition of the level of skill is generalized to more sophisticated games . The generalization for one-person games like Roulette, Blackjack and Golden Ten is discussed in section 4, for two-person zero-sum games like Chess, Schnapsen and two-person Poker in section 5, and, finally, for general n-person zero-sum games in section 6.

At this stage some general remarks about the differences in complexity between the various classes of games should be made. Casino games typically can be classified as zero-sum games: apart from possible admission fees money is reallocated among the participants (including the casino). Further, in a one-person game like Roulette each player's betting strategy is all-decisive upon the possible expected gains: these gains do not depend on the behaviour of other players. In a more-person game like Poker it is typically the case that a player's gains also depend on the opponents' behaviour. So, in particular, strategic aspects come in. For three- or more-person games also the feature of possible cooperation between subgroups of players should be taken into account. These aspects will be considered in detail in the various sections.
A numerical two-person variation of Poker will used to illustrate the underlying concepts in section 5, a three-person variation in section 6. With respect to the computational aspects concerning optimal strategies, we will make use of a modification of the learning-procedure as initially proposed by Brown (1949) and Robinson (1950) (cf. Shapiro (1958) and Karlin (1959)). This modification is discussed to some extent in section 5. Section 7 concludes with a discussion of implications of our measure of skill together with some ideas on future research.

2 Some qualitative criteria for skill

In our analysis we will restrict attention to games which, in principle, can be repeated under the same conditions. This assumption guarantees an objective quantification of uncertainty in terms of probability. This does not mean that such a quantification is impossible beforehand for games of a different kind but that it necessarily will be of a more subjective nature.

We formulate three requirements which in our opinion summarize the basic ideas underlying the Dutch legislation concerning the exploitation of games with chance elements.

R1. The WKS applies exclusively to situations which involve the exploitation of games with monetary prizes. For this reason we only consider games in which the "game-result" of a player can be expressed in some way in terms of a certain gain (or loss) of money.

R2. The skill of a player should be measured as his gain in the long run, i.e. in terms of expected gains. Necessary for a game of skill is that these expected gains vary among players.

R3. The fact that there is a difference between players with respect to their expected gains, does not immediately imply that the underlying game is a game of skill. Sufficient for a game of skill is that the chance elements involved do no prohibit these differences to be substantial.

In van der Genugten and Borm (1994, 1996) each of these requirements is discussed in detail in the light of the Dutch jurisprudence on specific games as Saturne, Blackjack, Roulette and Golden Ten. Implicitly these cases make clear that one should distinguish between three types of players:

(1) a beginner who plays the game in the naive way of somebody who has just mastered the rules of the game.

(2) an average player who can be thought to represent the vast majority of players.

(3) an advanced player who uses skill at the highest possible level and who sets an example for other players.

One may conclude that the WKS is concerned with the difference between a beginner and an average player. Therefore, we call the means of the difference in expected monetary gains (R1) between those two types of players (R2) the *learning* effect. This effect should be judged substantial in relation with the restrictive possibilities within the game set by the chance elements (R3) (e.g. with restrictions on the occurrence of maximal gains). In this judgement the advanced player has no direct role.

The aim of this paper is to define, for games within a broad class, the level of skill as a number S in the interval [0,1]. This number describes to which amount possible differences in expected gains between players are restricted by the chance elements within the game.

A *pure game of chance* will be game for which S equals 0, a *pure game of skill* a game with S equal to 1.

A qualification of a specific game as a game of chance or as a game of skill in the context of the WKS, will then depend on the bound the legislator sets within the interval [0,1], below which a game should be classified as a game of chance.

3 Basic games

In a basic game all participants face the same opponent which is called the bank. There is no strategic interaction between the players and the bank follows a predescribed and fixed strategy. Moreover, for each player the outcome of a basic game is one of the following two: either losing the stake-money or gaining a fixed multiple of the stake-money. According to the qualitative discussion of section 2 the following data should be sufficient to decide about the level of skill of a basic game:

p_0	:	the probability that a beginner wins
p_a	:	the probability that an average player wins
W_l	:	gains in case of a loss (i.e. minus the stake-money)
W_u	:	gains in case of a win (a multiple of the stake-money)

The difference $p_a - p_0$ provides information about the difference in skill between a beginner and an average player but does not reflect the consequences w.r.t. the monetary results as required by R2. Therefore we will take the average gains W_0 and W_a of a beginner and an average player, respectively, as a base for our analysis. Here, for a basic game,

$$W_0 := (1 - p_0)W_l + p_0 W_u \tag{3.1}$$

and

$$W_a := (1 - p_a)W_l + p_a W_u. \tag{3.2}$$

Clearly,

$$W_l \leq W_0 \leq W_a \leq W_u. \tag{3.3}$$

A relative measure for the difference in skill can be obtained by dividing the difference $W_a - W_0$ by $W_a - W_l$ which measures the difference in skill between an average player and a fictitious worst[3] player who, in a basic game, would always succeed in losing the stake-money. This leads to the quotient

[3] For a precise definition we refer to section 4.

$$Q_a := \frac{W_a - W_0}{W_a - W_l}. \tag{3.4}$$

This quotient satisfies the requirements R1 and R2, but it does not express the restrictive role of the chance elements as required by R3.
For this reason we will compare Q_a with a similar quotient Q_u in which the role of the average player is taken over by a fictitious best player who, in a basic game, would always succeed in winning:

$$Q_u := \frac{W_u - W_0}{W_u - W_l}. \tag{3.5}$$

In particular, due to the restrictive role of the chance elements an average player can only realize W_a in stead of W_u. Hence, a comparison between the relative measures Q_a and Q_u for the difference in skill w.r.t. a beginner does also justice to the requirement R3. This leads to the following definition of the level of skill S_a:

$$S_a := \frac{Q_a}{Q_u}. \tag{3.6}$$

In theory, expression (3.6) provides a method to rank basic games with respect to skill. This formula registers in a strictly defined way how the learning effect (R2) is restricted (R3) by the chance elements present in the game (roughly speaking: in comparison to maximal gains and maximal losses). The choice for the expression given in (3.6) may seem rather arbitrary but this is not quite true since a definition of the level of skill has to fulfil several logical requirements. Some of these requirements are loosely formulated below.

* A level of skill should not be dependent on the unity of money (e.g. dollars or pounds) in which gains and losses are expressed.
* In comparing two games with about the same maximal gains/losses, the game with the higher learning effect should get the higher level of skill.
* In comparing two games with about the same learning effect, the game with the higher maximal gains/losses should get the lower level of skill.
* A small change in the rules of the game should result in only a small change in the level of skill.

The formula given in (3.6) will not be the only one satisfying the requirements above and, possibly, a modification will be necessary if one asks for additional requirements. This, however, does not constitute a problem regarding the purpose for which this formula was developed.

In practice, however, difficulties will arise in determining W_0 and W_a since this requires specific assumptions on the behaviour of a beginner and an average player. Perhaps the less difficult problem of the two is how to characterize of a beginner. For example, in games with a structure like Roulette, choosing a purely random outcome seems a rather natural line of play for a beginner. Further, for games which are exploited in practice one can gain insight in how beginners behave just by means of observation. Moreover, a skilled analysis of the rules of the game might help from a normative point of view. In any case, for basic games, an exact specification of p_0 is not that important since the impact on the level of skill of a small change in p_0 is rather small. A more exact specification of p_a, however, is desirable.

How to determine the behaviour of an average player? A statistical approximation of p_a will be difficult to find since a precise definition of the population of average players is difficult to provide. Any particular choice of definition will be necessarily subjective and, further, practical implementation will be time-consuming and therefore costly. Moreover, for games which are not exploited (yet) a statistical approach is impossible.

To solve this problem we choose a relatively simple and pragmatic approach. It is a fact that (for finite zero-sum games) the expected gains of an *optimal*, i.e. most advanced, player are fully determined by the rules and structure of the game. So, in particular, these gains can be measured in an objective way.

Replacing the gains W_a of an average player by the gains W_m of an optimal player in expression(3.4) and subsequently also in (3.6) one obtains an objective and, in principle, operational criterium to determine the level of skill of a basic game.

More specifically, for a basic game, we define p_m as the probability that an optimal player wins and

$$W_m := (1 - p_m)W_l + p_m W_u \tag{3.7}$$

as the average gain of an optimal player.

Clearly it holds that

$$W_l \leq W_0 \leq W_a \leq W_m \leq W_u. \tag{3.8}$$

Further we define

$$Q_m := \frac{W_m - W_0}{W_m - W_l} \tag{3.9}$$

as a relative measure for the difference in skill between an optimal player and a beginner, and

$$S := \frac{Q_m}{Q_u} \tag{3.10}$$

as the level of skill of the underlying game.
Note that S is well-defined if the game is such that

$$W_l < W_m \text{ and } W_0 < W_u \tag{3.11}$$

which is an assumption only violated by trivial non-interesting games.

Since $\frac{W-W_0}{W-W_l}$ is increasing in W we have that $S \in[0,1]$ and also that $S \geq S_a$. This means that in going from S_a to S there is an upward shift in the level of skill. Since this is the case for *each* game separately, one may expect that replacing W_a by W_m will have little effect on the mutual ranking of games w.r.t. the level of skill. However, one should take this upward shift into account when determining an absolute bound on the level of skill below which a game should be considered as a game of chance.

Let us now consider the two extreme cases for the level of skill S. The case $S = 0$ boils down to $W_0 = W_m$ or, equivalently, to $p_0 = p_m$. This means that the probability of winning is the same for a beginner and a most advanced player. Consequently, the players can not exercise any influence on their chances. For this reason a basic game with $S = 0$ is called a pure game of chance. Analogously, a basic game with $S = 1$ is called a pure game of skill since this is equivalent to $W_m = W_u$ or, to $p_m = 1$. This means that the chance mechanism has no effect in the sense that an optimal player can always win.

As a numerical illustration we consider the following example.

Example 3.1

Consider the two basic games determined by the following data

	game I	game II
p_0	0.050	0.45
p_m	0.054	0.47
W_l	-10	-100
W_u	180	100

table 3.1

For game I the probabilities of winning are low but the relative gains of winning (18 times the stake-money) are high. For game II in some sense the opposite holds: the probabilities of winning are high but the relative gains are low.
Our analysis w.r.t. skill leads to results of table 3.2.

	game I	game II
W_l	-10	-100
W_0	-0.5	-10
W_m	0.26	-6
W_u	180	100
Q_m	0.07	0.04
Q_u	0.95	0.55
S	0.07	0.07

Table 3.2

So, in spite of the seemingly different structure of the two games, game I and II have about the same level of skill if one expresses the qualitative requirements (R1), (R2) and (R3) in a formal way.

4 One-person games

In this section we extend the definition of the level of skill from basic games towards one-person games like Roulette and Blackjack. In fact, basic games form a subclass of the one-person games: the bank follows a fixed and predescribed strategy and there is no interaction with the other participants in the game. However, basic games are restrictive in the sense that they allow only two outcomes: losing the stake-money or gaining a multiple of it.

In practice, even for relatively simple casino games, a player (or team) has to choose between various actions, possibly all with different monetary implications. For this reason the analysis of one-person games can not be based on the probabilities of winning for a beginner and an optimal player and on the gains in case of a loss and win, simply because these concepts can not be defined in an unambiguous way.

Instead we will start out directly from the following four notions of expected gains:

W_l	:	expected gains of a fictitious worst player
W_0	:	expected gains of a beginner
W_m	:	expected gains of an optimal player
W_u	:	expected gains of a fictitious best player

Here, a fictitious worst (best) player must be thought of as a player who gets to know the outcome of the chance elements before actually playing and tries to lose (gain) as much as possible.

Due to the rules of a basic game a fictitious worst (best) player will always succeed in losing (winning) and thus the notions of W_l and W_u indeed generalize the corresponding notions for basic games.

The expected gains W_0 and W_m are fully determined by a description of the strategy of a beginner and an optimal player, respectively.

For one-person games with

$$W_l < W_m \text{ and } W_0 < W_u \tag{4.1}$$

we define, in a similar way as for basic games, the quotients Q_m and Q_u by

$$Q_m := \frac{W_m - W_0}{W_m - W_l}, \quad Q_u := \frac{W_u - W_0}{W_u - W_l} \tag{4.2}$$

and the level of skill by

$$S := \frac{Q_m}{Q_u}. \tag{4.3}$$

We also adopt the same terminology: a one-person game with $S = 0$ is called a pure game of chance, with $S = 1$ a pure game of skill.

Example 4.1 (American Roulette)

In American Roulette a player can choose between a great variety of actions. The main choice however is between "simple" (e.g. rouge/noir, pair/impair) or not (e.g. plein, cheval, carré). For precise details also on the calculations below, we refer to van der Genugten and Borm (1994).
Suppose one chooses not to play "simple". Then, no matter what the precise action is, one's expected gains are $-\frac{1}{37}$, if we normalize the stake-money to 1.
Now suppose one chooses to play "simple". Then the winning outcomes never include "0", but if the outcome of a game is "0", one gets half of the state-money back. This leads to the expected gains of $-\frac{1}{74}$.
Consequently, an optimal player will choose "simple". Hence, $W_l = -1$ (a fictitious worst player will always succeed in losing the stake-money), $W_u = 35$ (knowing the outcome a fictitious best player will choose plein on this outcome) and $W_m = -\frac{1}{74}$.
What about W_0? One can discuss on the behaviour of a beginner in American Roulette. If one assumes that a beginner chooses to play simple, then $W_0 = -\frac{1}{74}$ and $S = 0$.
If one assumes that a beginner chooses not to play simple, then $W_0 = -\frac{1}{37}$.
By substitution it follows that

$$S = Q_m/Q_u = \frac{-\frac{1}{74} + \frac{1}{37}}{-\frac{1}{74} + 1} \bigg/ \frac{35 + \frac{1}{37}}{35 + 1} \approx 0.014\ .$$

We may conclude that American Roulette approximately is a pure game of chance.

Example 4.2 (Blackjack)
Blackjack as played in the Holland Casino's is extensively studied in van der Genugten (1993). Using the technique of simulation the following numbers were derived for the

case where the maximum stake is normalized to 1 and the minimum stake is 1 percent of the maximum:

$$\begin{aligned} W_l &= -0.89 \\ W_0 &= -0.056 \\ W_m &= 0.002 \\ W_u &= 0.54 \end{aligned}$$

The value of W_0 corresponds to the strategy "mimic the dealer". The values of W_l and W_u reflect the fact that the rules of the game are such that even if one knows the outcome of the chance mechanism i.e. one knows the precise order of the cards, one can not always succeed in winning or losing.
By substitution we find that the level of skill for Blackjack equals 0.16.
The Supreme Court in the Netherlands has explicitly qualified Blackjack as a game of chance. Hence, if one uses the above measure for the notion of skill, a consistency argument implies that each game with a level of skill below 0.16 should be considered as a game of chance w.r.t. Dutch legislation.

Example 4.3 (Golden Ten)
By means of a statistical analysis of data concerning the results of actual players (for details we refer to van der Genugten and Borm (1991)) it can be concluded that the level of skill for Golden Ten is somewhere between 0.20 and 0.30 depending on specific rules of the game.

5 Two-person zero-sum games

In a two-person game it is typically the case that the expected gains of a player depend not only on his own strategy choice but also on the strategy choice of the other player. In this section we will consider strictly competitive games in the sense that the gains of one player equal the losses of the other in each instance of the game. Put differently, the players exchange an amount of money specified by the outcome corresponding to one specific strategy-combination. This type of game is called a zero-sum game because the total gains of the players equal zero. For example, a Poker game as Seven Card Stud typically can be classified as a zero-sum game: apart from an admission fee, money is reallocated among the participants.

To study skill in games we will use an objective analysis of strategic behaviour (such as braging) based on the assumption of rational players. This approach follows the lines set out by *game theory* : a mathematical theory of conflict situations which started out from the pioneering work of von Neumann and Morgenstern (1944) and led quite recently the Nobel prize for economics for Nash, Harsanyi and Selten in 1994.

Consider an arbitrary two-person zero-sum game in which both player 1 and player 2 can choose from a finite number of actions. In this game we will allow a player to choose a mixed strategy, which provides a probability distribution on the set of actions.
Then this game has a uniquely determined (minimax-) *value* V in the sense that player 1 can choose a mixed strategy which guarantees him an expected payoff of at least V (independent of player 2's strategy choice), and player 2 has a mixed strategy which keeps his losses down to at most V. The strategies concerned are called minimax strategies and can be interpreted as describing optimal play.

Due to these facts we are able to extend the definition of the level of skill to an arbitrary (finite) zero-sum game with two players 1 and 2.
Assume we are in the role of player 1. If we fix a specific minimaxstrategy of player 2 in the sense that we assume that player 2 will act accordingly, the original two-person game can be reduced to a one-person game because player 1 faces a fixed but probabilistic environment, and the four basic notions of section 4 of expected gains (w.r.t. player 1) are well-defined:

$$W_l(1) \leq W_0(1) \leq W_m(1) \leq W_u(1). \tag{5.1}$$

Note that, by definition, $W_m(1)$ equals the value of the game.
Analogously, in the role of player 2, we assume a specific minimax strategy of player 1 to be given, which reduces the situation to a one-person game. This leads to the following notions of expected gains w.r.t. player 2:

$$W_l(2) \leq W_0(2) \leq W_m(2) \leq W_u(2), \tag{5.2}$$

where $W_m(2)$ equals minus the value of the game.

To define the overall notion of skill in the game it is reasonable to average between the role of player 1 and player 2, implicitly assuming a player of the game to take both the role of player 1 and player 2 every two instances of the game.
This leads to

$$\begin{array}{rcl} W_l & = & \frac{1}{2}(W_l(1) + W_l(2)) \\ W_0 & = & \frac{1}{2}(W_0(1) + W_0(2)) \\ W_m & = & \frac{1}{2}(W_m(1) + W_m(2)) \\ W_u & = & \frac{1}{2}(W_u(1) + W_u(2)) \end{array} \tag{5.3}$$

and, clearly,

$$W_l \leq W_0 \leq W_m \leq W_u. \tag{5.4}$$

Further note that, since $W_m(1) = -W_m(2)$

$$W_m = 0. \tag{5.5}$$

As before, we will assume that

$$W_l < W_m \text{ and } W_0 < W_u. \tag{5.6}$$

Subsequently, we define

$$Q_m = \frac{W_m - W_0}{W_m - W_l} \text{ and } Q_u = \frac{W_u - W_0}{W_u - W_l} \tag{5.7}$$

and

$$S = \frac{Q_m}{Q_u} \tag{5.8}$$

as the level of skill.

The case $S = 0$ is equivalent with $W_m(1) = W_0(1)$ and $W_m(2) = W_0(2)$ and $S = 1$ with $W_m(1) = W_u(1)$ and $W_m(2) = W_u(2)$. Accordingly, we can adopt the same terminology as before. A two-person zero-sum game with $S = 0$ is called a pure game of chance, while a game with $S = 1$ is refered to as a pure game of skill.

One final remark concerning the definition of the level of skill for two-person games is that it indeed generalizes the definition for one-person games.
Consider a one-person game with player 1 as its only participant who receives or pays money from or to a bankholder. This game can be viewed as a two-person zero-sum game in which the role of player 2 is taken by the bankholder. By definition of a one-person game, player 2 has no freedom in his choice of action and follows a predescribed strategy. Hence, $W_l(1), W_0(1), W_m(1)$ and $W_u(1)$ equal the corresponding one-person notions introduced in section 4 and the level of skill in the one-person framework (cf. (4.3)) is given by

$$S(1) = \frac{W_m(1) - W_0(1)}{W_m(1) - W_l(1)} \bigg/ \frac{W_u(1) - W_0(1)}{W_u(1) - W_l(1)} .$$

In the two-person framework we also have to analyze the role of player 2. However, for this analysis we assumed optimal behaviour of player 1 and since player 2 has no freedom of strategy choice it follows that

$$W_l(2) = W_0(2) = W_m(2) = W_u(2) = -W_m(1).$$

Hence,

$$W_l = \tfrac{1}{2}(W_l(1) - W_m(1)), W_0 = \tfrac{1}{2}(W_0(1) - W_m(1))$$
$$W_m = 0, W_u = \tfrac{1}{2}(W_u(1) - W_m(1))$$

and by substitution it follows that the level of skill S for the two-person game as defined by (5.8) equals $S(1)$.

Example 5.1 (Two-person Mini Poker)

Two-person Mini Poker is a game of cards played by two players, named player 1 and player 2, and with three cards of which only the numeric value is important. These values are 10, 20 and 30, respectively.
Before playing both players donate $ 1 to the stakes. After (re)shuffling the deck of cards each player is dealt one card. Each player knows (the value of) his own card but not the card of this opponent. Thus, the one card which remains in the deck is not shown to either of the players.
Player 1 starts the play and has to decide between "checking" or "raising". If he decides to check, player 2 has to check too and "showdown" follows. If player 1 decides to raise, he has to add one extra dollar to the stakes. Subsequently, player 2 has to decide between "folding" or "calling". If the decides to fold, player 1 gets the stakes. If player 2 decides to raise, he also has to add one extra dollar to the stakes and "showdown" follows. If the players have decided upon "showdown" both cards are compared and the player with the highest card value gets the stakes.

In this simple game of Poker both chance (dealing cards) and skill (w.r.t. a good betting strategy) play a role. We will reach the conclusion that the level of skill S equals about 0.28.

First we will describe the possible strategies of both players. For each of the three possible cards, both player 1 and player 2 have to choose between two possible actions, leading to a total of 8 combinations. The so-called pure strategies are represented in table 5.1 and table 5.2.

	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\underline{6}$	$\underline{7}$	$\underline{8}$
10:	C	C	C	C	R	R	R	R
20:	C	C	R	R	C	C	R	R
30:	C	R	C	R	C	R	C	R

Table 5.1: Pure strategies of player 1.

	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\underline{6}$	$\underline{7}$	$\underline{8}$
10:	F	F	F	F	C	C	C	C
20:	F	F	C	C	F	F	C	C
30:	F	C	F	C	F	C	F	C

Table 5.2: Pure strategies of player 2.

For example, the strategy $\underline{4}$ of player 1 should be interpreted as

"Check with 10, Raise with 20, Raise with 30",

and strategy $\underline{6}$ of player 2 as

"Call with 10, Fold with 20, Call with 30",

if you are called upon to act.

Allowing for mixed strategies, each player may put a probability measure on his set of pure strategies. Mixed strategies will be denoted in the following way: the strategy $\left(\frac{1}{2}\underline{2}, \frac{1}{2}\underline{4}\right)$ of player 1 will indicate that player 1 will choose the pure strategy $\underline{2}$ with probability $\frac{1}{2}$ and the pure strategy $\underline{4}$ with probability $\frac{1}{2}$. In the equivalent behavioral interpretation this boils down to:

"Check with 10, Raise with 30, and with 20:

Check with probability $\frac{1}{2}$ and Raise with probability $\frac{1}{2}$".

In analysing the level of skill of this game of Poker we first need the strategy of a beginner in both the role of player 1 and player 2.

W.r.t. player 1 we find it reasonable for a naive player to check with 10 and to raise with 30. How to act with 20? Probably it is wise to vary between checking and raising, and using symmetry arguments, to check and raise with equal probability. In fact, the

same reasoning applies w.r.t. player 2. If he is called upon to act, we assume he folds with 10, calls with 30 and folds and calls with equal probability with 20.
Using the notation introduced above, this boils down to the beginner's strategy $\left(\frac{1}{2}\underline{2}, \frac{1}{2}\underline{4}\right)$ for both roles.

The next step is to determine minimaxstrategies for both player 1 and player 2. To this aim we first calculate the 8×8 matrix which describes the expected playoff to player 1 for each possible pure strategy combination. The result is presented in table 5.3.

player 2	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\underline{6}$	$\underline{7}$	$\underline{8}$
player 1								
$\underline{1}$	0	0	0	0	0	0	0	0
$\underline{2}$	0	0	1/6	1/6	1/6	1/6	1/3	1/3
$\underline{3}$	1/3	−1/6	1/3	−1/6	1/2	0	1/2	0
$\underline{4}$	1/3	−1/6	1/2	0	2/3	1/6	5/6	1/3
$\underline{5}$	2/3	1/6	1/6	−1/3	2/3	1/6	1/6	−1/3
$\underline{6}$	2/3	1/6	1/3	−1/6	5/6	1/3	1/2	0
$\underline{7}$	1	0	1/2	−1/2	7/6	1/6	2/3	−1/3
$\underline{8}$	1	0	2/3	−1/3	4/3	1/3	1	0

Table 5.3: Two-person Mini Poker as a matrix game

More specifically, in table 5.3 the result of a specific strategy combination is a fair average over the outcomes w.r.t. the six possible card combinations.
By iteratively deleting dominated strategies (e.g. strategy $\underline{1}$ of player 1 is dominated by strategy $\underline{2}$), the 8×8 matrix game of table 5.3 can be reduced to the simple 2×2 matrix game represented in table 5.4.

player 1	$\underline{2}$	$\underline{4}$
player 2		
$\underline{2}$	0	1/6
$\underline{6}$	1/6	−1/6

Table 5.4: The reduced game

From table 5.4 it is readily derived that the value of the game is $\frac{1}{18}$ and the minimaxstrategies of the players are uniquely determined:

$\left(\frac{2}{3}\underline{2}, \frac{1}{3}\underline{6}\right)$ for player 1 and $\left(\frac{2}{3}\underline{2}, \frac{1}{3}\underline{4}\right)$ for player 2.

Note that $\left(\frac{2}{3}\underline{2}, \frac{1}{3}\underline{6}\right)$ indeed guarantees a payoff of $\frac{1}{18}$ because this strategy leads to an expected payoff of at least $\frac{1}{18}$ against every pure strategy of player 2 (see table 5.5) and hence also against every mixed strategy of player 2.

player 2 / player 1	$\underline{1}$	$\underline{2}$	$\underline{3}$	$\underline{4}$	$\underline{5}$	$\underline{6}$	$\underline{7}$	$\underline{8}$
$\left(\frac{2}{3}\underline{2}, \frac{1}{3}\underline{6}\right)$	4/18	1/18	4/18	1/18	7/18	4/18	7/18	4/18

Table 5.5: Payoffs resulting from the minimax strategy of player 1.

Similarly, one can check that the strategy $\left(\frac{2}{3}\underline{2}, \frac{1}{3}\underline{4}\right)$ of player 2 guarantees this player an expected loss of at most $\frac{1}{18}$.

For determining $W_l(1), W_0(1), W_m(1)$ and $W_u(1)$ we assume that player 2 behaves according to his unique optimal strategy $\left(\frac{2}{3}\underline{2}, \frac{1}{3}\underline{4}\right)$.

Since we have chosen $\left(\frac{1}{2}\underline{2}, \frac{1}{2}\underline{4}\right)$ as a beginner's strategy for player 1, it follows from table 5.3 that

$$W_0(1) = \frac{1}{2} \cdot \frac{2}{3} \cdot (0) + \frac{1}{2} \cdot \frac{1}{3} \cdot \left(\frac{1}{6}\right) + \frac{1}{2} \cdot \frac{2}{3} \cdot \left(-\frac{1}{6}\right) + \frac{1}{2} \cdot \frac{1}{3} \cdot (0) = -\frac{1}{36}.$$

Further, since $W_m(1)$ equals the value of the game, we have that

$$W_m(1) = \frac{1}{18}.$$

To determine $W_l(1)$ and $W_u(1)$, we have to consider a fictitious worst and a fictitious best player in the role of player 1, respectively.

By assumption, a fictitious worst (best) player will know the precise card combination (C_1, C_2), with $C_i \in \{10, 20, 30\}, C_1 \neq C_2$, denoting the value of the card of player i, before having to decide upon checking or raising.

Moreover, since we fixed the strategy of player 2 to $\left(\frac{2}{3}\underline{2}, \frac{1}{3}\underline{4}\right)$ one can calculate for each card combination what the expected gains of the two possible actions will be. For each card combination a fictitious worst (best) player will subsequently choose the action with minimal (maximal) gains. The numbers $W_l(1)$ and $W_u(1)$ then will be the fair average of the six corresponding gains. This computation, which is illustrated in table 5.6, leads to

$$W_l(1) = -\frac{1}{3} \text{ and } W_u(1) = \frac{2}{9}.$$

	player 2: $\left(\frac{2}{3}\underline{2}, \frac{1}{3}\underline{4}\right)$						
card comb.	(10, 20)	(10, 30)	(20, 10)	(20, 30)	(30, 10)	(30, 20)	
Check Raise	-1 0	-1 -2	1 1	-1 -2	1 1	1 $\frac{4}{3}$	
Min Max	-1 0	-2 -1	1 1	-2 -1	1 1	1 $\frac{4}{3}$	$W_l(1) = -\frac{1}{3}$ $W_u(1) = \frac{2}{9}$

Table 5.6: The calculation of $W_l(1)$ and $W_u(1)$.

In an analogous way one derives

$$W_l(2) = -\frac{5}{9}, W_0(2) = -\frac{1}{18}, W_m(2) = -\frac{1}{18} \text{ and } W_u(2) = \frac{1}{9}.$$

Hence,

$$W_l = -\frac{4}{9}, W_0 = -\frac{1}{24}, W_m = 0, W_u = \frac{1}{6}$$

and, by substitution, the level of skill equals

$$S = 0.275.$$

Example 5.1 is rather special in the sense that the minimaxstrategies of both player 1 and player 2 are uniquely determined. If this is not the case, the analysis requires a specific choice between all possible minimax strategies of a player. The minimaxstrategies we select will be called *learnminimax strategies*. They have a strong intuitive appeal because they are the outcomes of (a modification of) learning procedures as initially proposed by Brown (1949) and Robinson (1950), and elaborated on by Shapiro (1958) and Karlin (1959).

The following procedure describes how the learnminimaxstrategy $\hat{q}$ of player 2 is determined.

Step 1

(a) Let $p_1 = \bar{p}_1$ be the strategy of a beginner in the role of player 1.

(b) Determine the set Y_1 of pure strategies of player 2 which are optimal (i.e. best replies) against p_1. Let $\bar{q}_1$ be the mixed strategy of player 2 that puts equal probability on the elements of Y_1. Define $q_1 := \bar{q}_1$ as the first step learn minimax strategy of player 2.

Step 2

(a) Determine the set X_2 of pure strategies of player 1 which are optimal against q_1. Let $\bar{p}_2$ be the mixed strategy of player 1 that puts equal probability on the elements of X_2. Define $p_2 := \left(\frac{1}{2}\bar{p}_1, \frac{1}{2}\bar{p}_2\right)$, using obvious notation.

(b) Determine the set Y_2 of pure strategies of player 2 which are optimal against p_2. Let $\bar{q}_2$ be the mixed strategy of player 2 that puts equal probability on the elements of Y_2. Define $q_2 := \left(\frac{1}{2}\bar{q}_1, \frac{1}{2}\bar{q}_2\right)$ as the second step learnminimax strategy of player 2.

Step n

(a) Define the set X_n of pure strategies of player 1 which are optimal against q_{n-1}. Let $\bar{p}_n$ be the mixed strategy of player 1 that puts equal probability on the elements of X_n. Define $p_n := \left(\frac{1}{n}\bar{p}_1, \frac{1}{n}\bar{p}_2, \ldots, \frac{1}{n}\bar{p}_n\right)$.

(b) Determine the set Y_n of pure strategies of player 2 which are optimal against p_n. Let $\bar{q}_n$ be the mixed strategy of player 2 that puts equal probability on the elements of Y_n. Define $q_n := \left(\frac{1}{n}\bar{q}_n, \frac{1}{n}\bar{q}_2, \ldots, \frac{1}{n}\bar{q}_n\right)$ as the n-th step learnminimax strategy of player 2.

Denoting the (minimax) value of the underlying game by V and the payoff to player 1

w.r.t. the strategy combination (p_n, q_n) by V_n, it holds that $\lim_{n\to\infty} V_n = V$. (cf. Robinson (1950)).
The learnminimaxstrategy $\hat{q}$ of player 2 is then defined by

$$\hat{q} := \lim_{n\to\infty} q_n. \tag{5.9}$$

In an analogous way, starting out from the strategy of a beginner in the role of player 2, the learnminimax strategy $\hat{p}$ of player 1 can be determined.

Example 5.2
Reconsider the game of example 5.1. Suppose however that the game is not played with 3 cards, with values 10, 20 and 30, but with $n \geq 4$ cards, with values 10, 20, 30, ...,10.n instead. Then, using the concept of learnminimax strategies, the game can be analyzed along the same lines as in example 5.1.
For $n = 4, 5, \ldots, 10$ the results are given in table 5.7

n	4	5	6	7	8	9	10
S	0.22	0.09	0.16	0.16	0.16	0.19	0.20

Table 5.7: The level of skill of two-person Mini-Poker with n cards.

6 n-Person Zero-Sum Games

For a two-person zero-sum game the players are direct opposites in the sense that the gains of one player equal the losses of the other in each instance of the game.
Although in each instance of an n-person zero-sum game with $n \geq 3$ money is reallocated among the players, two particular participants can not be viewed as direct opposites in the sense that they should act in a way to oppose each other in the strongest sense. The mutual competition is of a more indirect and complex nature. Moreover, it is conceivable that subgroups of players will (possibly implicitly) cooperate during certain stages of the game to oppose the other participants in a more effective way. Put in game theoretic terms this means that coalitions (alliances) between players are formed. The situation is even more complex since coalitions might change during one instance of the game.

Since our aim is to measure the impact of chance elements in a game, we will not analyze an n-person game in all its strategic and cooperative details. To determine the level of skill for this type of games in a consistent way we make the assumption that in each instance of the game player i evaluates the situation as if he faces only one opponent: the coalition of all other participants which coordinates actions in such a way to oppose player i in the strongest sense.

These considerations boil down to the following definitions w.r.t. the level of skill of an arbitrary zero-sum game with player set $N = \{1, 2, \ldots, n\}$ with $n \geq 3$.

Assume one is in the role of player $i \in N$. By assumption we consider the two-person zero-sum game between player i and the coalition $N \backslash \{i\}$ in our analysis of skill. If we fix a specific minimax strategy of "player" $N \backslash \{i\}$ in this two-person game, a one-person situation is obtained and the following four basic notions of expected gains of player i are well-defined:

$$W_l(i) \leq W_0(i) \leq W_m(i) \leq W_u(i) \tag{6.1}$$

As before, if the minimax strategy of player $N \backslash \{i\}$ is not uniquely determined, then we will use the learnminimax strategy of player $N \backslash \{i\}$ w.r.t. the strategy of a beginner in the role of player i.

To define the general notion of skill, we average between the possible roles a participant can have:

$$W_l = \frac{1}{n}\sum_{i\in N} W_l(i), W_0 = \frac{1}{n}\sum_{i\in N} W_0(i), W_m = \frac{1}{n}\sum_{i\in N} W_m(i), W_u = \frac{1}{n}\sum_{i\in N} W_u(i) \quad (6.2)$$

and, clearly,

$$W_l \le W_0 \le W_m \le W_u. \quad (6.3)$$

As before, we will assume that

$$W_l < W_m \text{ and } W_0 < W_u. \quad (6.4)$$

Subsequently, we define

$$Q_m = \frac{W_m - W_0}{W_m - W_l}, \quad Q_u = \frac{W_u - W_0}{W_u - W_l} \quad (6.5)$$

and

$$S = \frac{Q_m}{Q_u} \quad (6.6)$$

as the level of skill.

Note that this definition of the level of skill also can be adopted if $n = 2$. Obviously, this renders the same definition as provided in section 5. Finally, the case $S = 0$ is equivalent with $W_m(i) = W_0(i)$ for all $i \in N$, and the case $S = 1$ with $W_m(i) = W_u(i)$ for all $i \in N$. Therefore, an n-person zero-sum game with $S = 0$ is called a pure game of chance while a game with $S = 1$ is called a pure game of skill.

Example 6.1 (Three-person Mini Poker)

Three-person Mini Poker is a game of cards played by three players 1, 2 and 3 and four cards with numerical values 10, 20, 30 and 40.

The rules are similar to the rules of two-person Mini Poker. After donating 1$ to the stakes each player is dealt one card. Player 1 starts the play and has to decide between checking or raising. If he checks, player 2 and 3 have to check too and showdown follows.

If he raises, then first player 2 has to decide between folding or calling, after which player 3 has to make the same choice. If a player raises or calls, he has to add one extra dollar to the stakes. If both player 2 and 3 fold the stakes go to player 1. Otherwise, showdown between the two or three players left follows. If showdown ocours, the player with the highest (value of his) card gets the stakes.

We will not analyze this game in all its details but only highlight some important aspects. In principle player 1 can choose between $16 = 2^4$ possible (pure) strategies: for each of his four possible cards he can choose between checking or raising. For the strategy of a beginner in the role of player 1 we take the pure strategy of Table 6.1.

Card of 1	10	20	30	40
Action	C	C	R	R

Table 6.1: Player 1's strategy as a beginner
(C = Check, R = Raise)

Player 2 is only called to play if player 1 raises. In that case he has to decide between folding or calling. So, in principle, also player 2 can choose between $16 = 2^4$ possible pure strategies because his decision might depend on his card.

For a beginner in the role of player we take the choices of table 6.2.

Card of 2	10	20	30	40
Action	F	F	C	C

Table 6.2: Player 2's strategy as a beginner
(F = Fold, C = Call)

For player 3 the situation is a bit more complicated in the sense that he could let his choice between folding or calling depend not only on his card but also on the action taken by player 2. This implies a total of $256 = 2^8$ pure strategies. For a beginner in the role of player 3 we take the strategy represented in table 6.3.

Card of 3	10	10	20	20	30	30	40	40
Observed action of 2	F	C	F	C	F	C	F	C
Action of 3	F	F	F	F	C	C	C	C

Table 6.3: Player 3's strategy as a beginner
(F = Fold, C = Call)

Note that in the game of player 1 against the coalition $\{2,3\}$ which we use in our analysis of skill player 1 can choose between the same 16 pure strategies, while the coalition $\{2,3\}$ can choose between 4696 (=16 . 256) pure strategies. For the corresponding two-person game it turns out, that the (coordinated) minimax strategy of "player" $\{2,3\}$ is essentially unique and given in Table 6.4 (with obvious notation).

Card of 2	10	20	30	40
Action of 2	F	F	($\frac{3}{4}$ F, $\frac{1}{4}$ C)	C

Card of 3	10	10	20	20	30	30	40	40
Observed action of 2	F	C	F	C	F	C	F	C
Action of 3	F	F	F	F	($\frac{3}{4}$ F, $\frac{1}{4}$ C)	F or C	C	C or F

Table 6.4: The minimaxstrategy of coalition $\{2,3\}$.

In this two-person game there is also a unique minimaxstrategy for player 1: see table 6.5.

Card of 1	10	20	30	40
Action of 1	C	C	($\frac{3}{4}$ C, $\frac{1}{4}$ R)	R

Table 6.5: The minimaxstrategy of player 1.

The value of this game is $\frac{1}{24}$. Following a similar analysis as in example 5.1 we find
$W_l(1) = -0.500$, $W_0(1) = -0.125$, $W_m(1) = 0.041$ and $W_u(1) = 0.375$.

Moreover, proceeding along the same lines and using learnminimaxstrategies (for details we refer to van der Genugten and Borm (1994))

$$W_l(2) = -0.573, \quad W_0(2) = -0.028, \quad W_m(2) = -0.028, \quad W_u(2) = 0.140$$
$$W_l(3) = -0.294, \quad W_0(3) = -0.041, \quad W_m(3) = -0.021, \quad W_u(3) = 0.063$$

and, consequently

$$W_l = -0.456,\ W_0 = -0.064,\ W_m = -0.002 \text{ and } W_u = 0.193.$$

Hence, for the level of skill S we find that

$$S = 0.35.$$

7 A classification of games

In this section we present a global classification of some well-known round games w.r.t. the level of skill. The corresponding order is partly based on the (objective) analysis introduced in this paper and partly on a (more subjective) comparison between the various possibilities of exercising skill in the underlying games.
The main results are summarized in the following overview.

Pure games of chance [0]
Roulette, Craps, Trente et Quarante [0.0]
Blackjack [0.16]
Golden Ten [0.20]
Schnapsen
Draw Poker
Texas Hold'Em
Seven Card Stud
Bridge
Chess, Checkers
Pure games of skill [1]

The ordering between the three variants of Poker is based on the following considerations. In principle all variants allow a rational player to exercise skill in two ways:

- with knowledge on the statistical probabilities w.r.t. the final hands of the players,
- with knowledge on a controlled strategy of braging.

In Draw Poker, the second aspect is present in a prominent way but the first aspect comes forward only in the relatively simple form of evaluating the possibilities of your own hand without knowing any other cards. In both Texas Hold'Em and Seven Card Stud the aspect of braging is less prominent but the first-mentioned aspect of counting and estimating probabilities is essential (in Seven Card Stud to a larger extent than in Texas Hold'Em) to raise one's expected gains.

Interestingly, Dutch jurisprudence has classified Golden Ten as a game of chance while according to Austrian jurisprudence Schnapsen is a game of skill. If we combine these judgements, the decision bound between games of chance and games of skill should be laid between Golden Ten and Schnapsen.

Of course, the classification given above should not be interpreted as the final order in which no shifts are possible. A further analysis, in particular w.r.t. more sophisticated variants of Poker than the ones discussed in section 5 and 6, is needed to further sharpen and quantify the classification.

References

1. Brown, G.W. (1949). *Some notes on computation of game solutions.* RAND Report P.78, The RAND Corporation, Santa Monica, California.
2. Epstein, R.A. (1977). *The theory of gambling and statistical logic.* Academic Press, New York.
3. Van der Genugten, B.B. (1993). *Blackjack in Holland Casino's: hoe de dealer te verslaan!* Tilburg University Press, Tilburg.
4. Van der Genugten, B.B. and Borm, P. (1991). *How to distinguish between games of chance and games of skill: an application to Golden Ten* (In Dutch). Report, Department of Econometrics, Tilburg University.
5. Van der Genugten, B.B. and Borm, P. (1994). *A comparison between games with chance elements w.r.t. skill* (In Dutch). Report, Department of Econometrics, Tilburg University.
6. Van der Genugten, B.B. and Borm, P. (1996). *On the level of skill and the diagnostic criteria* (In Ducth). Report, Department of Econometrics, Tilburg University.
7. Karlin, S. (1959). *Matrix games, Programming and Mathematical Economics*, vol. I, Pergamon Press.
8. Neumann, J. von and Morgenstern, O. (1944). *Theory of games and economic behavior.* Princeton University Press, Princeton.
9. Robinson, J. (1950). "An iterative method of solving a game". *Annals of Mathematics*, 54, 296-301.
10. Scarne, J. (1990). *Guide to modern Poker.* Simon and Schuster, New York.

11. Shapiro, H.N. (1958). "Note on a computation method in the theory of games". *Communications on Pure and Applied Mathematics*, 11, 587-593.

12. Tamburin, H. (1993). *References guide to casino gambling.* Research Service Ltd. Mobile.